

Dynamic Monopolies for Degree Proportional Thresholds in Connected Graphs of Girth at least Five and Trees

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Abstract

Let G be a graph, and let $\rho \in [0, 1]$. For a set D of vertices of G , let the set $H_\rho(D)$ arise by starting with the set D , and iteratively adding further vertices u to the current set if they have at least $\lceil \rho d_G(u) \rceil$ neighbors in it. If $H_\rho(D)$ contains all vertices of G , then D is known as an irreversible dynamic monopoly or a perfect target set associated with the threshold function $u \mapsto \lceil \rho d_G(u) \rceil$. Let $h_\rho(G)$ be the minimum cardinality of such an irreversible dynamic monopoly.

For a connected graph G of maximum degree at least $\frac{1}{\rho}$, Chang (Triggering cascades on undirected connected graphs, Information Processing Letters 111 (2011) 973-978) showed $h_\rho(G) \leq 5.83\rho n(G)$, which was improved by Chang and Lyuu (Triggering cascades on strongly connected directed graphs, Theoretical Computer Science 593 (2015) 62-69) to $h_\rho(G) \leq 4.92\rho n(G)$. We show that for every $\epsilon > 0$, there is some $\rho(\epsilon) > 0$ such that $h_\rho(G) \leq (2 + \epsilon)\rho n(G)$ for every ρ in $(0, \rho(\epsilon))$, and every connected graph G that has maximum degree at least $\frac{1}{\rho}$ and girth at least 5. Furthermore, we show that $h_\rho(T) \leq \rho n(T)$ for every ρ in $(0, 1]$, and every tree T that has order at least $\frac{1}{\rho}$.

Keywords: Irreversible dynamic monopoly; perfect target set
MSC2010: 05C69

1 Introduction

We consider finite, simple, and undirected graphs, and use standard terminology and notation.

Let G be a graph with vertex set $V(G)$. Let $\phi : V(G) \rightarrow \mathbb{N}_0$ be a threshold function such that $\phi(u)$ is at most the degree $d_G(u)$ of u in G for every vertex u of G . For a set D of vertices of G , let $H_{(G,\phi)}(D)$ be the smallest set \bar{D} of vertices of G such that $D \subseteq \bar{D}$, and every vertex u in $V(G) \setminus \bar{D}$ has less than $\phi(u)$ neighbors in \bar{D} . Note that the set $H_{(G,\phi)}(D)$ can be constructed by starting with the set D , and iteratively adding further vertices u to the current set if they have at least $\phi(u)$ neighbors in it. Such iterative expansion processes have been considered in a variety of contexts [2, 6–9, 13, 15, 18, 21]. If $H_{(G,\phi)}(D) = V(G)$, then D is a ϕ -dynamic monopoly of G . Let $h_\phi(G)$ be the minimum cardinality of a ϕ -dynamic monopoly of G .

By a simple probabilistic argument, very similar to the one used by Alon and Spencer [4] to prove the Caro-Wei bound on the independence number of a graph [6, 20], Ackerman, Ben-Zwi, Wolfowitz [1] showed

$$h_\phi(G) \leq \sum_{u \in V(G)} \frac{\phi(u)}{d_G(u) + 1}. \quad (1)$$

Essentially the same bound was obtained independently by Reichman [19]. For an application of this argument to independence in hypergraphs see [5].

It is easy to see [9] that (minimum) dynamic monopolies and (maximum) generalized degenerate induced subgraphs are dual notions, that is, (1) can be considered the dual counterpart of bounds as in [3].

A very natural choice for the threshold function is to assign values that are proportional to the vertex degrees. Specifically, for some real parameter ρ in $[0, 1]$, let

$$\phi_\rho : V(G) \rightarrow \mathbb{N}_0 : u \mapsto \lceil \rho d_G(u) \rceil.$$

If D is a random set of vertices of G that contains each vertex independently at random with probability ρ , then, for every vertex u of G , the expected number of neighbors of u that belong to D is $\rho d_G(u)$. This suggests that $h_{\phi_\rho}(G)$ might be only slightly bigger than $\rho n(G)$, where $n(G)$ is the order of G . Without any restriction on ρ or G though, this intuition is misleading. In fact, if ρ is positive but much smaller than $\frac{1}{n(G)}$, then $h_{\phi_\rho}(G)$ is at least 1, while $\rho n(G)$ can be arbitrarily small. As observed by Chang [10], it is reasonable to consider only values of ρ that are at least $\frac{1}{\Delta(G)}$, where $\Delta(G)$ is the maximum degree of G , because $\phi_{\frac{1}{\Delta(G)}} = \phi_\rho$ for every ρ in $\left(0, \frac{1}{\Delta(G)}\right]$. For a connected graph G and $\rho \in \left[\frac{1}{\Delta(G)}, 1\right]$, Chang [10] proved

$$h_{\phi_\rho}(G) \leq (2\sqrt{2} + 3)\rho n(G) \approx 5.83\rho n(G), \quad (2)$$

which was improved by Chang and Lyuu [11] to

$$h_{\phi_\rho}(G) \leq 4.92\rho n(G). \quad (3)$$

Note that the bound in (1) might evaluate to $\Omega(n(G))$, because, for instance, vertices of degree 1 contribute $\frac{1}{2}$ rather than $O(\rho)$ to the right hand side of (1). In fact, especially for small values of ρ , and graphs with many vertices of small degrees, the bound (3) can be much better than the bound (1).

The proof strategies for (2) and (3) are quite different. The bound (2) is proved by a suitable adaptation of the argument of Ackerman, Ben-Zwi, Wolfowitz [1]. Vertices of small degree, that is, at most $\frac{1}{\rho}$, are treated differently from those of large degree, that is, more than $\frac{1}{\rho}$. A small set X_0 of vertices of large degree ensures that the remaining vertices of large degree have few neighbors of small degree outside of $H_{(G, \phi_\rho)}(X_0)$. This allows to apply the argument of Ackerman et al. to the vertices of large degree outside of X_0 . The bound (3) is proved by a random procedure that considers a sequence X_1, X_2, \dots of random sets of vertices each containing every individual vertex independently at random with probability 3.51ρ . Starting with the empty set, a ϕ_ρ -dynamic monopoly is constructed by iteratively adding the vertices in $X_i \setminus H_{(G, \phi_\rho)}(X_1 \cup \dots \cup X_{i-1})$ to the current set. Chernoff's inequality is used to ensure that $H_{(G, \phi_\rho)}(X_1 \cup \dots \cup X_i)$ grows sufficiently fast. The proof of (3) has some resemblance to iterative random procedures that are used to show lower bounds on the independence number [14, 16].

It is a natural to ask for the best-possible constant in bounds of the form (2) and (3). We contribute to this question by showing the following results.

Theorem 1 *For every positive real ϵ , there is some positive real $\rho(\epsilon)$ such that*

$$h_{\phi_\rho}(G) \leq (2 + \epsilon)\rho n(G) \quad (4)$$

for every ρ in $(0, \rho(\epsilon))$, and every connected graph G that has maximum degree at least $\frac{1}{\rho}$ and girth at least 5.

The proof of Theorem 1 is based on a combination of the techniques from [10, 11]. Note that (5) requires a sufficiently small value of ρ , but that bounds like (2), (3), and (5) are especially interesting for small values of ρ . It is possible to generalize (5) to strongly connected directed graphs similarly as in [11].

Theorem 2 *If ρ is in $(0, 1]$, and T is a tree of order at least $\frac{1}{\rho}$, then*

$$h_{\phi_\rho}(T) \leq \rho n(T). \quad (5)$$

Note that $h_{\phi_\rho}(T)$ can be computed in linear time [7] for a given tree.

For many more references to and discussion of related work see [10, 11].

2 Proofs of Theorem 1 and Theorem 2

We recall two tools from probability theory.

- **Markov's inequality** (cf. Theorem 3.2 in [17])

If X is a non-negative random variable, and $t > 0$, then $\mathbb{P}[X \geq t] \leq \frac{1}{t}\mathbb{E}[X]$.

- **Chernoff's inequality** (cf. Theorem 4.2 in [17])

If Z_1, \dots, Z_n are independent random variables, and p_1, \dots, p_n in $(0, 1)$ are such that $\mathbb{P}[Z_i = 1] = p_i$ and $\mathbb{P}[Z_i = 0] = 1 - p_i$ for $i \in [n]$, then

$$\mathbb{P}\left[Z_1 + \dots + Z_n < (1 - \delta)\mathbb{E}[Z_1 + \dots + Z_n]\right] < e^{-\frac{\delta^2}{2}\mathbb{E}[Z_1 + \dots + Z_n]}$$

for every $0 < \delta \leq 1$.

We proceed to the first proof.

Proof of Theorem 1: Let $\epsilon > 0$. Let $\delta > 0$ be small enough such that

$$\delta \leq \min\left\{e^{-\frac{1}{4}}, \frac{1}{2}\right\}, \text{ and} \quad (6)$$

$$(1 + \delta)^2 + \frac{1 + \delta}{(1 - \delta)^2} \leq 2 + \epsilon. \quad (7)$$

Let $\rho(\epsilon) > 0$ be small enough such that

$$\rho(\epsilon) \leq \frac{\delta}{1 + \delta} \left(1 - e^{-\frac{\delta^2}{2(1 - \delta)}}\right) \frac{1}{8 \ln\left(\frac{1}{\delta}\right)}. \quad (8)$$

Let $\rho \in (0, \rho(\epsilon))$. Note that (8) implies $\rho \leq \delta$. Let G be a connected graph of maximum degree at least $\frac{1}{\rho}$ and girth at least 5.

Let

$$\begin{aligned} V_1 &= \left\{u \in V(G) : d_G(u) < \frac{1}{\rho}\right\}, \text{ and} \\ V_2 &= V(G) \setminus V_1. \end{aligned}$$

Note that $\phi_\rho(u) = 1$ for $u \in V_1$, and $V_2 \neq \emptyset$.

Throughout this proof, we write ' $H(X)$ ' instead of ' $H_{(G, \phi_\rho)}(X)$ '.

Claim 1 *There is a set $X_0 \subseteq V_2$ with the following properties.*

(i) $|N_G(u) \cap (V_1 \setminus H(X_0))| \leq \frac{1}{1+\delta} d_G(u)$ for every $u \in V_2 \setminus X_0$.

(ii) $|X_0| \leq (1+\delta)\rho n(G)$.

Proof: Let (x_1, \dots, x_k) be a maximal sequence of distinct vertices from V_2 such that

$$\left| N_G(x_i) \cap \left(V_1 \setminus H(\{x_j : 1 \leq j \leq i-1\}) \right) \right| > \frac{1}{1+\delta} d_G(x_i)$$

for $1 \leq i \leq k$.

Let $X_0 = \{x_1, \dots, x_k\}$. The maximality of the sequence (x_1, \dots, x_k) implies (i).

Since $\phi_\rho(u) = 1$ for $u \in V_1$, and $d_G(x_i) \geq \frac{1}{\rho}$ for $1 \leq i \leq k$, we have

$$\begin{aligned} |H(\{x_j : 1 \leq j \leq i\})| - |H(\{x_j : 1 \leq j \leq i-1\})| &\geq \left| N_G(x_i) \cap \left(V_1 \setminus H(\{x_j : 1 \leq j \leq i-1\}) \right) \right| \\ &> \frac{1}{1+\delta} d_G(x_i) \\ &\geq \frac{1}{(1+\delta)\rho}, \end{aligned}$$

which implies $\frac{k}{(1+\delta)\rho} < |H(\{x_1, \dots, x_k\})| \leq n(G)$. It follows that $|X_0| = k \leq (1+\delta)\rho n(G)$, which completes the proof of the claim. \square

For $i \in \mathbb{N}$, let X_i be a random subset of $V(G) \setminus H(X_0)$ that contains each vertex of $V(G) \setminus H(X_0)$ independently at random with probability

$$p^{(1)} = \frac{\rho}{(1-\delta)}. \quad (9)$$

Note that $\rho \leq \delta$ implies $p^{(1)} \leq \frac{\delta}{(1-\delta)} \stackrel{(6)}{\leq} 1$. Furthermore, note that X_0 is chosen deterministically, and that the random sets X_1, X_2, \dots are chosen independently of each other.

Let $Y_0 = X_0$, and, for $i \in \mathbb{N}$, let

$$\begin{aligned} Y_i &= X_i \setminus H(Y_0 \cup \dots \cup Y_{i-1}), \text{ and} \\ Y_{\leq i} &= Y_0 \cup Y_1 \cup \dots \cup Y_i. \end{aligned}$$

By construction, $X_0 \cup X_1 \cup \dots \cup X_i \subseteq H(Y_{\leq i-1}) \cup Y_i \subseteq H(Y_{\leq i})$, which implies

$$H(X_0 \cup X_1 \cup \dots \cup X_i) = H(Y_{\leq i}).$$

Claim 2 $\mathbb{P}[u \notin H(X_0 \cup X_i)] \leq \delta$ for $i \in \mathbb{N}$ and $u \in V_2 \setminus H(X_0)$.

Proof: Let

$$\begin{aligned} N_1 &= N_G(u) \cap (V_1 \setminus H(X_0)), \\ N_2 &= N_G(u) \setminus (V_1 \cup H(X_0)), \text{ and} \\ N_3 &= N_G(u) \cap H(X_0). \end{aligned}$$

By Claim 1, $|N_1| \leq \frac{1}{1+\delta} d_G(u)$, which implies

$$|N_2| + |N_3| = d_G(u) - |N_1| \geq \frac{\delta}{1+\delta} d_G(u) \stackrel{u \in V_2}{\geq} \frac{\delta}{(1+\delta)\rho}. \quad (10)$$

Let G_u be the subgraph of G that arises from the induced subgraph

$$G \left[N_G(u) \cup \bigcup_{v \in N_2} (N_G(v) \setminus \{u\}) \right]$$

by removing, for every vertex v in N_2 and every vertex w in $N_G(v) \setminus \{u\}$, every edge incident with w except for the edge vw . Since G has girth at least 5, all vertices in $N_1 \cup N_3$ are isolated in G_u , and each vertex v in N_2 belongs to a component of G_u that is a star of order $d_G(v)$ with center v whose set of endvertices is $N_G(v) \setminus \{u\}$.

Let

$$H_u = H_{(G_u, \phi_\rho)} \left((H(X_0) \cup X_i) \cap V(G_u) \right).$$

Since G_u is a subgraph of G , we have $H_u \subseteq H(H(X_0) \cup X_i) = H(X_0 \cup X_i)$.

The structure of G_u and the choice of X_i implies that

- every vertex in N_1 belongs to H_u with probability $p^{(1)}$, and
- every vertex in N_3 belongs to H_u with probability 1.

Let $v \in N_2$. Note that $v \notin H_u$ implies $v \notin X_i$ as well as $|(N_G(v) \setminus \{u\}) \cap X_i| < \rho d_G(v)$, that is, $|(N_G[v] \setminus \{u\}) \cap X_i| < \rho d_G(v)$, where $N_G[v] = \{v\} \cup N_G(v)$. Since $|N_G[v] \setminus \{u\}| = d_G(v)$, the definitions of $p^{(1)}$ and X_i imply

$$\begin{aligned} \mathbb{P}[v \notin H_u] &\leq \mathbb{P}[|(N_G[v] \setminus \{u\}) \cap X_i| < \rho d_G(v)] \\ &\stackrel{(9)}{=} \mathbb{P}[|(N_G[v] \setminus \{u\}) \cap X_i| < (1 - \delta)p^{(1)}d_G(v)] \\ &= \mathbb{P}[|(N_G[v] \setminus \{u\}) \cap X_i| < (1 - \delta)\mathbb{E}[|(N_G[v] \setminus \{u\}) \cap X_i|]] \\ &< e^{-\delta^2 \mathbb{E}[|(N_G[v] \setminus \{u\}) \cap X_i|]/2}, \end{aligned}$$

where the last inequality is a consequence of Chernoff's inequality.

Since

$$\mathbb{E}[|(N_G[v] \setminus \{u\}) \cap X_i|] = p^{(1)}d_G(v) \stackrel{(9)}{=} \frac{\rho d_G(v)}{(1 - \delta)} \stackrel{v \in V_2}{\geq} \frac{1}{(1 - \delta)},$$

we obtain

$$\mathbb{P}[v \notin H_u] < e^{-\frac{\delta^2}{2(1-\delta)}}.$$

This implies that

- every vertex in $N_2 \cup N_3$ belongs to H_u with probability more than $p^{(2)}$, where

$$p^{(2)} = 1 - e^{-\frac{\delta^2}{2(1-\delta)}}. \tag{11}$$

By the structure of G_u , the events $[v \in H_u]$ are all independent for $v \in N_G(u)$. Let X_u be a random subset of $N_2 \cup N_3$ that contains each vertex of $N_2 \cup N_3$ independently at random with probability $p^{(2)}$.

Note that

$$\mathbb{E}[|(N_2 \cup N_3) \cap X_u|] = p^{(2)}(|N_2| + |N_3|) \stackrel{(10)}{\geq} \frac{\delta p^{(2)}}{(1 + \delta)\rho} \stackrel{(8)}{\geq} 8 \ln \left(\frac{1}{\delta} \right), \tag{12}$$

and

$$\frac{\rho d_G(u)}{p^{(2)}(|N_2| + |N_3|)} \stackrel{(10)}{\leq} \frac{\rho(1+\delta)}{\delta p^{(2)}} \stackrel{(8)}{\leq} \frac{1}{8 \ln(\frac{1}{\delta})} \stackrel{(6)}{\leq} \frac{1}{2}. \quad (13)$$

Since $u \notin H(X_0 \cup X_i)$ implies $|(N_2 \cup N_3) \cap H_u| < \rho d_G(u)$, the definition of X_u and Chernoff's inequality imply

$$\begin{aligned} \mathbb{P}[u \notin H(X_0 \cup X_i)] &\leq \mathbb{P}[|(N_2 \cup N_3) \cap H_u| < \rho d_G(u)] \\ &\leq \mathbb{P}[|(N_2 \cup N_3) \cap X_u| < \rho d_G(u)] \\ &= \mathbb{P}\left[|(N_2 \cup N_3) \cap X_u| < \frac{\rho d_G(u)}{p^{(2)}(|N_2| + |N_3|)} p^{(2)}(|N_2| + |N_3|)\right] \\ &\stackrel{(12),(13)}{\leq} \mathbb{P}\left[|(N_2 \cup N_3) \cap X_u| < \frac{1}{2} \mathbb{E}[|(N_2 \cup N_3) \cap X_u|]\right] \\ &< e^{-\frac{1}{8}} \mathbb{E}[|(N_2 \cup N_3) \cap X_u|] \\ &\stackrel{(12)}{\leq} \delta. \end{aligned}$$

□

Claim 3 $\mathbb{P}[u \notin H(Y_{\leq i}) \mid u \notin H(Y_{\leq i-1})] \leq \delta$ for $i \in \mathbb{N}$ and $u \in V(G) \setminus H(X_0)$.

Proof: Recall that the set X_0 is chosen deterministically, that is, the only source of randomness are the sets X_1, X_2, \dots , which are chosen independently of each other. This implies that for every two not necessarily distinct vertices u' and u'' of $V(G)$, the two events $[u' \notin H(X_0 \cup X_i)]$ and $[u'' \notin H(Y_{\leq i-1})]$ are independent.

First, let $u \in V_2$. Since $H(X_0 \cup X_i) \subseteq H(Y_{\leq i})$, Claim 2 and the independence observed above imply

$$\begin{aligned} \mathbb{P}[u \notin H(Y_{\leq i}) \mid u \notin H(Y_{\leq i-1})] &\leq \mathbb{P}[u \notin H(X_0 \cup X_i) \mid u \notin H(Y_{\leq i-1})] \\ &= \mathbb{P}[u \notin H(X_0 \cup X_i)] \\ &\leq \delta. \end{aligned}$$

Next, let $u \in V_1$. Let $u_0 \dots u_\ell$ be a shortest path in G between $u = u_0$ and some vertex u_ℓ in V_2 . Since $\phi_\rho(u_0) = \dots = \phi_\rho(u_{\ell-1}) = 1$, we obtain that $u \notin H(Y_{\leq i})$ implies $u_\ell \notin H(Y_{\leq i})$, which implies $u_\ell \notin H(X_0 \cup X_i)$. Now, Claim 2 and the independence observed above imply

$$\begin{aligned} \mathbb{P}[u \notin H(Y_{\leq i}) \mid u \notin H(Y_{\leq i-1})] &\leq \mathbb{P}[u_\ell \notin H(Y_{\leq i}) \mid u \notin H(Y_{\leq i-1})] \\ &\leq \mathbb{P}[u_\ell \notin H(X_0 \cup X_i) \mid u \notin H(Y_{\leq i-1})] \\ &= \mathbb{P}[u_\ell \notin H(X_0 \cup X_i)] \\ &\leq \delta. \end{aligned}$$

□

Note that $u \notin H(Y_{\leq i})$ implies $u \notin H(Y_{\leq i-1})$, and that $\mathbb{P}[u \notin H(Y_{\leq i})] = 0$ for $u \in H(X_0)$. By Claim 3, for $i \in \mathbb{N}$, we obtain, by linearity of expectation,

$$\begin{aligned} \mathbb{E}[|V(G) \setminus H(Y_{\leq i})|] &= \sum_{u \in V(G) \setminus H(X_0)} \mathbb{P}[u \notin H(Y_{\leq i})] \\ &= \sum_{u \in V(G) \setminus H(X_0)} \mathbb{P}[u \notin H(Y_{\leq i}) \mid u \notin H(Y_{\leq i-1})] \mathbb{P}[u \notin H(Y_{\leq i-1})] \\ &\leq \delta \sum_{u \in V(G) \setminus H(X_0)} \mathbb{P}[u \notin H(Y_{\leq i-1})] \\ &= \delta \mathbb{E}[|V(G) \setminus H(Y_{\leq i-1})|], \end{aligned} \quad (14)$$

which is equivalent to

$$\mathbb{E}[|V(G) \setminus H(Y_{\leq i-1})|] \leq \frac{1}{1-\delta} (\mathbb{E}[|H(Y_{\leq i})|] - \mathbb{E}[|H(Y_{\leq i-1})|]). \quad (15)$$

Iteratively applying (14), we obtain

$$\mathbb{E}[|V(G) \setminus H(Y_{\leq i})|] \leq \delta^i n(G).$$

Furthermore, by the choice of X_i and the definition of Y_i , we have $\mathbb{P}[u \in Y_i \mid u \notin H(Y_{\leq i-1})] = p^{(1)}$ and $\mathbb{P}[u \in Y_i \mid u \in H(Y_{\leq i-1})] = 0$, which implies

$$\begin{aligned} \mathbb{E}[|Y_i|] &= \sum_{u \in V(G) \setminus H(X_0)} \mathbb{P}[u \in Y_i] \\ &= \sum_{u \in V(G) \setminus H(X_0)} \mathbb{P}[u \in Y_i \mid u \notin H(Y_{\leq i-1})] \mathbb{P}[u \notin H(Y_{\leq i-1})] \\ &= p^{(1)} \sum_{u \in V(G) \setminus H(X_0)} \mathbb{P}[u \notin H(Y_{\leq i-1})] \\ &= p^{(1)} \mathbb{E}[|V(G) \setminus H(Y_{\leq i-1})|] \\ &\stackrel{(15)}{\leq} \frac{p^{(1)}}{1-\delta} (\mathbb{E}[|H(Y_{\leq i})|] - \mathbb{E}[|H(Y_{\leq i-1})|]). \end{aligned}$$

Therefore, by Claim 1,

$$\begin{aligned} \mathbb{E}[|Y_{\leq i}|] &= \mathbb{E}[|Y_0 \cup Y_1 \cup \dots \cup Y_i|] \\ &\leq |X_0| + \mathbb{E}[|Y_1 \cup \dots \cup Y_i|] \\ &\leq |X_0| + \frac{p^{(1)}}{1-\delta} \sum_{j=1}^i (\mathbb{E}[|H(Y_{\leq j})|] - \mathbb{E}[|H(Y_{\leq j-1})|]) \\ &\leq |X_0| + \frac{p^{(1)}}{1-\delta} n(G) \\ &\stackrel{(9)}{=} |X_0| + \frac{\rho}{(1-\delta)^2} n(G) \\ &\leq \rho(1+\delta)n(G) + \frac{\rho}{(1-\delta)^2} n(G) \\ &= \left((1+\delta) + \frac{1}{(1-\delta)^2} \right) \rho n(G). \end{aligned}$$

Let $k \in \mathbb{N}$ be large enough such that $\delta^k n(G) + \frac{1}{1+\delta} < 1$.

By Markov's inequality,

$$\begin{aligned} &\mathbb{P} \left[(|V(G) \setminus H(Y_{\leq k})| < 1) \wedge (|Y_{\leq k}| < (1+\delta)\mathbb{E}[|Y_{\leq k}|]) \right] \\ &\geq 1 - \mathbb{P}[|V(G) \setminus H(Y_{\leq k})| \geq 1] - \mathbb{P}[|Y_{\leq k}| \geq (1+\delta)\mathbb{E}[|Y_{\leq k}|]] \\ &\geq 1 - \delta^k n(G) - \frac{1}{1+\delta} \\ &> 0. \end{aligned}$$

By the first moment method [4], this implies the existence of a set D of vertices such that

$$\begin{aligned} |D| &\leq (1+\delta)\mathbb{E}[|Y_{\leq k}|] \\ &\leq (1+\delta) \left((1+\delta) + \frac{1}{(1-\delta)^2} \right) \rho n(G) \\ &\stackrel{(7)}{\leq} (2+\epsilon)\rho n(G), \end{aligned}$$

and $H(D) = V(G)$, which completes the proof of Theorem 1. \square

It is conceivable that (1) remains true without the girth condition as well as for arbitrary values of ρ in $\left[\frac{1}{\Delta(G)}, 1\right]$ and sufficiently large girth.

We proceed to the second proof.

Proof of Theorem 2: Let ρ be in $(0, 1]$. Suppose that the tree T is a counterexample of minimum order, that is, $n(T) \geq \frac{1}{\rho}$ and $h_{\phi_\rho}(T) > \rho n(T)$. Let $V_2 = \left\{u \in V(T) : d_T(u) \geq \frac{1}{\rho}\right\}$. Note that V_2 is a ϕ_ρ -dynamic monopoly of T . If $|V_2| \leq 1$, then $h_{\phi_\rho}(T) \leq |V_2| \leq \rho n(T)$, which is a contradiction. Hence, $|V_2| \geq 2$.

Let u in V_2 be chosen such that a largest component K of $T - u$ that contains a vertex from V_2 has largest possible order. Suppose that $T - u$ has a second component K' distinct from K that contains a vertex from V_2 . Let u' be in $V(K') \cap V_2$. Since $T - u'$ has a component that contains all vertices in $\{u\} \cup V(K)$, we obtain a contradiction to the choice of u . Hence, K is the only component of $T - u$ that contains a vertex from V_2 . Let $R = T - V(K)$. Since K and R both contain a vertex from V_2 , we have $n(K) \geq \frac{1}{\rho}$ and $n(R) \geq \frac{1}{\rho}$. By the choice of T , the tree K has a ϕ_ρ -dynamic monopoly D with $|D| \leq \rho n(K)$. Let v be the neighbor of u in K . Since $\rho d_T(v) \leq 1 + \rho(d_T(v) - 1) = 1 + \rho d_K(v)$, the set $\{u\} \cup D$ is a ϕ_ρ -dynamic monopoly of T of order at most $|D| + 1 \leq \rho n(K) + 1 \leq \rho n(K) + \rho n(R) = \rho n(T)$. This contradiction completes the proof. \square

If $\rho \in (0, 1]$ is such that $\frac{1}{\rho}$ is an integer, then $K_{1, \frac{1}{\rho}-1}$ shows that the bound in Theorem 2 is tight.

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